

A NEW GENERALIZATION OF SOME INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

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ABSTRACT. In this paper, a new identity for convex functions is derived. A consequence of the identity is that we can derive new estimates for the remainder term of the midpoint, trapezoid, and Simpson formulae for functions whose derivatives in absolute value at certain power are convex. Some applications to special means of real numbers are also given.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. See [1, 2, 3, 4], the results of the generalization, improvement and extension of the famous integral inequality (1.1).

The following inequality is well known in the literature as Simpson's inequality .

Let $f : [a, b] \rightarrow \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\|f^{(4)}\|_{\infty} = \sup_{x \in (a,b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:

$$\left| \frac{1}{3} \left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b-a)^2.$$

In recent years many authors have studied error estimations for Simpson's inequality; for refinements, counterparts, generalizations and new Simpson's type inequalities, see [5, 6, 7, 8]

In [8], Sarikaya et al. obtained inequalities for differentiable convex mapping which are connected Simpson's inequality, and they used the following lemma to prove this.

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Lemma 1. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be an absolutely continuous mapping on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. Then the following equality holds:

$$\begin{aligned} & \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \\ &= \frac{b-a}{2} \int_0^1 \left[\left(\frac{t}{2} - \frac{1}{3} \right) f' \left(\frac{1+t}{2}b + \frac{1-t}{2}a \right) + \left(\frac{1}{3} - \frac{t}{2} \right) f' \left(\frac{1+t}{2}a + \frac{1-t}{2}b \right) \right] dt. \end{aligned}$$

The main inequality in [8], pointed out, is as follows.

Theorem 1. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$, $q > 1$, then the following inequality holds,

$$\begin{aligned} (1.2) \quad & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{12} \left(\frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \left\{ \left(\frac{3|f'(b)|^q + |f'(a)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} \right\}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In [7], Sarikaya et al. obtained a new upper bound for the right-hand side of Simpson's inequality for convex mapping:

Corollary 1. Let $f : I \subset [0, \infty) \rightarrow \mathbb{R}$ be a differentiable mapping on I° , such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$, $q > 1$, then the following inequality holds,

$$\begin{aligned} (1.3) \quad & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{12} \left(\frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \left\{ \left(\frac{|f'(b)|^q + |f'(\frac{a+b}{2})|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^q + |f'(\frac{a+b}{2})|^q}{2} \right)^{\frac{1}{q}} \right\}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In [3], some inequalities of Hermite-Hadamard type for differentiable convex mappings were presented as follows.

Theorem 2. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds,

$$(1.4) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)| + |f'(b)|}{2} \right).$$

In this paper, in order to provide a unified approach to establish midpoint inequality, trapezoid inequality and Simpson's inequality for functions whose derivatives in absolute value at certain power are convex, we derive a general integral identity for convex functions. Finally some applications for special means of real numbers are provided.

2. MAIN RESULTS

In order to prove our main theorems, we need the following Lemma.

Lemma 2. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. Then the following equality holds:*

$$\begin{aligned}
 (2.1) \quad & \lambda(\alpha f(a) + (1 - \alpha)f(b)) + (1 - \lambda)f(\alpha a + (1 - \alpha)b) - \frac{1}{b - a} \int_a^b f(x) dx \\
 &= (b - a) \left[\int_0^{1-\alpha} (t - \alpha\lambda) f'(tb + (1 - t)a) dt \right. \\
 & \quad \left. + \int_{1-\alpha}^1 (t - 1 + \lambda(1 - \alpha)) f'(tb + (1 - t)a) dt \right].
 \end{aligned}$$

Proof. We note that

$$I = \int_0^{1-\alpha} (t - \alpha\lambda) f'(tb + (1 - t)a) dt + \int_{1-\alpha}^1 (t - 1 + \lambda(1 - \alpha)) f'(tb + (1 - t)a) dt$$

integrating by parts, we get

$$\begin{aligned}
 I &= (t - \alpha\lambda) \frac{f(tb + (1 - t)a)}{b - a} \Big|_0^{1-\alpha} - \int_0^{1-\alpha} \frac{f(tb + (1 - t)a)}{b - a} dt \\
 & \quad + (t - 1 + \lambda(1 - \alpha)) \frac{f(tb + (1 - t)a)}{b - a} \Big|_{1-\alpha}^1 - \int_{1-\alpha}^1 \frac{f(tb + (1 - t)a)}{b - a} dt \\
 &= (1 - \alpha - \alpha\lambda) \frac{f((1 - \alpha)b + \alpha a)}{b - a} + \frac{\alpha\lambda f(a)}{b - a} + \frac{(1 - \alpha)\lambda f(b)}{b - a} \\
 & \quad - (-\alpha + \lambda(1 - \alpha)) \frac{f((1 - \alpha)b + \alpha a)}{b - a} - \int_0^1 \frac{f(tb + (1 - t)a)}{b - a} dt.
 \end{aligned}$$

Setting $x = tb + (1 - t)a$, and $dx = (b - a) dt$, we obtain

$$(b - a)I = \lambda(\alpha f(a) + (1 - \alpha)f(b)) + (1 - \lambda)f(\alpha a + (1 - \alpha)b) - \frac{1}{b - a} \int_a^b f(x) dx$$

which gives the desired representation (2.1). \square

Theorem 3. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. If $|f'|^q$ is convex on $[a, b]$,*

$q \geq 1$, then the following inequality holds:

$$(2.2) \quad \left| \lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b - a} \int_a^b f(x) dx \right|$$

$$\leq \begin{cases} (b - a) \left\{ \gamma_2^{1 - \frac{1}{q}} (\mu_1 |f'(b)|^q + \mu_2 |f'(a)|^q)^{\frac{1}{q}} \right. \\ \quad \left. + v_2^{1 - \frac{1}{q}} (\eta_3 |f'(b)|^q + \eta_4 |f'(a)|^q)^{\frac{1}{q}} \right\}, & \alpha \lambda \leq 1 - \alpha \leq 1 - \lambda(1 - \alpha) \\ (b - a) \left\{ \gamma_2^{1 - \frac{1}{q}} (\mu_1 |f'(b)|^q + \mu_2 |f'(a)|^q)^{\frac{1}{q}} \right. \\ \quad \left. + v_1^{1 - \frac{1}{q}} (\eta_1 |f'(b)|^q + \eta_2 |f'(a)|^q)^{\frac{1}{q}} \right\}, & \alpha \lambda \leq 1 - \lambda(1 - \alpha) \leq 1 - \alpha \\ (b - a) \left\{ \gamma_1^{1 - \frac{1}{q}} (\mu_3 |f'(b)|^q + \mu_4 |f'(a)|^q)^{\frac{1}{q}} \right. \\ \quad \left. + v_2^{1 - \frac{1}{q}} (\eta_3 |f'(b)|^q + \eta_4 |f'(a)|^q)^{\frac{1}{q}} \right\}, & 1 - \alpha \leq \alpha \lambda \leq 1 - \lambda(1 - \alpha) \end{cases}$$

where

$$(2.3) \quad \gamma_1 = (1 - \alpha) \left[\alpha \lambda - \frac{(1 - \alpha)}{2} \right], \quad \gamma_2 = (\alpha \lambda)^2 - \gamma_1,$$

$$(2.4) \quad \begin{aligned} v_1 &= \frac{1 - (1 - \alpha)^2}{2} - \alpha [1 - \lambda(1 - \alpha)], \\ v_2 &= \frac{1 + (1 - \alpha)^2}{2} - (\lambda + 1)(1 - \alpha) [1 - \lambda(1 - \alpha)], \end{aligned}$$

$$(2.5) \quad \begin{aligned} \mu_1 &= \frac{(\alpha \lambda)^3 + (1 - \alpha)^3}{3} - \alpha \lambda \frac{(1 - \alpha)^2}{2}, \\ \mu_2 &= \frac{1 + \alpha^3 + (1 - \alpha \lambda)^3}{3} - \frac{(1 - \alpha \lambda)}{2} (1 + \alpha^2), \\ \mu_3 &= \alpha \lambda \frac{(1 - \alpha)^2}{2} - \frac{(1 - \alpha)^3}{3}, \\ \mu_4 &= \frac{(\alpha \lambda - 1)(1 - \alpha^2)}{2} + \frac{1 - \alpha^3}{3}, \end{aligned}$$

$$(2.6) \quad \begin{aligned} \eta_1 &= \frac{1 - (1 - \alpha)^3}{3} - \frac{[1 - \lambda(1 - \alpha)]}{2} \alpha (2 - \alpha), \\ \eta_2 &= \frac{\lambda(1 - \alpha) \alpha^2}{2} - \frac{\alpha^3}{3}, \end{aligned}$$

$$\begin{aligned} \eta_3 &= \frac{[1 - \lambda(1 - \alpha)]^3}{3} - \frac{[1 - \lambda(1 - \alpha)]}{2} \left(1 + (1 - \alpha)^2 \right) + \frac{1 + (1 - \alpha)^3}{3}, \\ \eta_4 &= \frac{[\lambda(1 - \alpha)]^3}{3} - \frac{\lambda(1 - \alpha) \alpha^2}{2} + \frac{\alpha^3}{3}. \end{aligned}$$

Proof. Suppose that $q \geq 1$. From Lemma 2 and using the well known power mean inequality, we have

$$\begin{aligned}
 & \left| \lambda (\alpha f(a) + (1 - \alpha) f(b)) + (1 - \lambda) f(\alpha a + (1 - \alpha) b) - \frac{1}{b - a} \int_a^b f(x) dx \right| \\
 & \leq (b - a) \left[\int_0^{1-\alpha} |t - \alpha\lambda| |f'(tb + (1 - t)a)| dt + \int_{1-\alpha}^1 |t - 1 + \lambda(1 - \alpha)| |f'(tb + (1 - t)a)| dt \right] \\
 & \leq (b - a) \left\{ \left(\int_0^{1-\alpha} |t - \alpha\lambda| dt \right)^{1-\frac{1}{q}} \left(\int_0^{1-\alpha} |t - \alpha\lambda| |f'(tb + (1 - t)a)|^q dt \right)^{\frac{1}{q}} \right. \\
 (2.7) \quad & \left. + \left(\int_{1-\alpha}^1 |t - 1 + \lambda(1 - \alpha)| dt \right)^{1-\frac{1}{q}} \left(\int_{1-\alpha}^1 |t - 1 + \lambda(1 - \alpha)| |f'(tb + (1 - t)a)|^q dt \right)^{\frac{1}{q}} \right\}.
 \end{aligned}$$

Since $|f'|^q$ is convex on $[a, b]$, we know that for $t \in [0, 1]$

$$|f'(tb + (1 - t)a)|^q \leq t |f'(b)|^q + (1 - t) |f'(a)|^q,$$

hence, by simple computation

$$\begin{aligned}
 (2.8) \quad & \int_0^{1-\alpha} |t - \alpha\lambda| dt = \begin{cases} \gamma_2, & \alpha\lambda \leq 1 - \alpha \\ \gamma_1, & \alpha\lambda \geq 1 - \alpha \end{cases}, \\
 & \gamma_1 = (1 - \alpha) \left[\alpha\lambda - \frac{(1 - \alpha)}{2} \right], \quad \gamma_2 = (\alpha\lambda)^2 - \gamma_1,
 \end{aligned}$$

$$(2.9) \quad \int_{1-\alpha}^1 |t - 1 + \lambda(1 - \alpha)| dt = \begin{cases} v_1, & 1 - \lambda(1 - \alpha) \leq 1 - \alpha \\ v_2, & 1 - \lambda(1 - \alpha) \geq 1 - \alpha \end{cases},$$

$$\begin{aligned}
 v_1 &= \frac{1 - (1 - \alpha)^2}{2} - \alpha [1 - \lambda(1 - \alpha)], \\
 v_2 &= \frac{1 + (1 - \alpha)^2}{2} - (\lambda + 1)(1 - \alpha) [1 - \lambda(1 - \alpha)],
 \end{aligned}$$

$$\begin{aligned}
 (2.10) \quad & \int_0^{1-\alpha} |t - \alpha\lambda| |f'(tb + (1 - t)a)|^q dt \leq \int_0^{1-\alpha} |t - \alpha\lambda| [t |f'(b)|^q + (1 - t) |f'(a)|^q] dt \\
 & = \begin{cases} \mu_1 |f'(b)|^q + \mu_2 |f'(a)|^q, & \alpha\lambda \leq 1 - \alpha \\ \mu_3 |f'(b)|^q + \mu_4 |f'(a)|^q, & \alpha\lambda \geq 1 - \alpha \end{cases},
 \end{aligned}$$

$$\begin{aligned}
 \mu_1 &= \frac{(\alpha\lambda)^3 + (1 - \alpha)^3}{3} - \alpha\lambda \frac{(1 - \alpha)^2}{2}, \\
 \mu_2 &= \frac{1 + \alpha^3 + (1 - \alpha\lambda)^3}{3} - \frac{(1 - \alpha\lambda)}{2} (1 + \alpha^2),
 \end{aligned}$$

$$\begin{aligned}\mu_3 &= \alpha\lambda\frac{(1-\alpha)^2}{2} - \frac{(1-\alpha)^3}{3}, \\ \mu_4 &= \frac{(\alpha\lambda-1)(1-\alpha^2)}{2} + \frac{1-\alpha^3}{3},\end{aligned}$$

and

$$\begin{aligned}(2.11) \quad & \int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)| |f'(tb+(1-t)a)|^q dt \\ & \leq \int_{1-\alpha}^1 |t-1+\lambda(1-\alpha)| [t|f'(b)|^q + (1-t)|f'(a)|^q] dt \\ & = \begin{cases} \eta_1 |f'(b)|^q + \eta_2 |f'(a)|^q, & 1-\lambda(1-\alpha) \leq 1-\alpha \\ \eta_3 |f'(b)|^q + \eta_4 |f'(a)|^q, & 1-\lambda(1-\alpha) \geq 1-\alpha \end{cases},\end{aligned}$$

where η_1, η_2, η_3 and η_4 are defined as in (2.6). Thus, using (2.8)-(2.11) in (2.7), we obtain the inequality (2.2). This completes the proof. \square

Corollary 2. *Let the assumptions of Theorem 3 hold. Then for $q = 1$ the inequality (2.2) reduced to the following inequality*

$$\begin{aligned}(2.12) \quad & \left| \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ & \leq \begin{cases} (b-a) \{(\mu_1 + \eta_3)|f'(b)| + (\mu_2 + \eta_4)|f'(a)|\}, & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ (b-a) \{(\mu_1 + \eta_1)|f'(b)| + (\mu_2 + \eta_2)|f'(a)|\}, & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ (b-a) \{(\mu_3 + \eta_3)|f'(b)| + (\mu_4 + \eta_4)|f'(a)|\}, & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{cases}\end{aligned}$$

Corollary 3. *Let the assumptions of Theorem 3 hold. Then for $\alpha = \frac{1}{2}$ and $\lambda = \frac{1}{3}$, from the inequality (2.2) we get the following Simpson type inequality*

$$\begin{aligned}(2.13) \quad & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ & \leq (b-a) \left(\frac{5}{72} \right)^{1-\frac{1}{q}} \left\{ \left(\frac{29}{1296} |f'(b)|^q + \frac{61}{1296} |f'(a)|^q \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\frac{61}{1296} |f'(b)|^q + \frac{29}{1296} |f'(a)|^q \right)^{\frac{1}{q}} \right\},\end{aligned}$$

which is the same of the inequality in [7, Theorem 10] for $s = 1$.

Corollary 4. *Let the assumptions of Theorem 3 hold. Then for $\alpha = \frac{1}{2}$ and $\lambda = 0$, from the inequality (2.2) we get the following midpoint type inequality*

$$\begin{aligned}(2.14) \quad & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ & \leq \frac{b-a}{8} \left\{ \left(\frac{|f'(b)|^q + 2|f'(a)|^q}{3} \right)^{\frac{1}{q}} + \left(\frac{2|f'(b)|^q + |f'(a)|^q}{3} \right)^{\frac{1}{q}} \right\}\end{aligned}$$

Corollary 5. *In Corollary 4, if $q = 1$, then we have the following midpoint type inequality*

$$(2.15) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)| + |f'(b)|}{2} \right).$$

which is the same of the inequality (1.4).

Corollary 6. *Let the assumptions of Theorem 3 hold. Then for $\alpha = \frac{1}{2}$ and $\lambda = 1$, from the inequality (2.2) we get the following trapezoid type inequality*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ & \leq \frac{b-a}{8} \left\{ \left(\frac{|f'(b)|^q + 5|f'(a)|^q}{6} \right)^{\frac{1}{q}} + \left(\frac{5|f'(b)|^q + |f'(a)|^q}{6} \right)^{\frac{1}{q}} \right\} \end{aligned}$$

Using Lemma 2 we shall give another result for convex functions as follows.

Theorem 4. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. If $|f'|^q$ is convex on $[a, b]$, $q > 1$, then the following inequality holds:*

$$(2.16) \quad \left| \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq (b-a)$$

$$\times \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \begin{aligned} & \left[(1-\alpha)^{\frac{1}{q}} \varepsilon_1^{\frac{1}{p}} \delta_1^{\frac{1}{q}} + \alpha^{\frac{1}{q}} \varepsilon_3^{\frac{1}{p}} \delta_2^{\frac{1}{q}} \right], \quad \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ & \left[(1-\alpha)^{\frac{1}{q}} \varepsilon_1^{\frac{1}{p}} \delta_1^{\frac{1}{q}} + \alpha^{\frac{1}{q}} \varepsilon_4^{\frac{1}{p}} \delta_2^{\frac{1}{q}} \right], \quad \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ & \left[(1-\alpha)^{\frac{1}{q}} \varepsilon_2^{\frac{1}{p}} \delta_1^{\frac{1}{q}} + \alpha^{\frac{1}{q}} \varepsilon_3^{\frac{1}{p}} \delta_2^{\frac{1}{q}} \right], \quad 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{aligned} \right.$$

where

$$(2.17) \quad \delta_1 = \frac{|f'((1-\alpha)b + \alpha a)|^q + |f'(a)|^q}{2}, \quad \delta_2 = \frac{|f'((1-\alpha)b + \alpha a)|^q + |f'(b)|^q}{2},$$

$$\begin{aligned} \varepsilon_1 &= (\alpha\lambda)^{p+1} + (1-\alpha-\alpha\lambda)^{p+1}, \quad \varepsilon_2 = (\alpha\lambda)^{p+1} - (\alpha\lambda - 1 + \alpha)^{p+1}, \\ \varepsilon_3 &= [\lambda(1-\alpha)]^{p+1} + [\alpha - \lambda(1-\alpha)]^{p+1}, \quad \varepsilon_4 = [\lambda(1-\alpha)]^{p+1} - [\lambda(1-\alpha) - \alpha]^{p+1}, \end{aligned}$$

and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2 and by Hölder's integral inequality, we have

$$\begin{aligned}
& \left| \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq (b-a) \left[\int_0^{1-\alpha} |t - \alpha\lambda| |f'(tb + (1-t)a)| dt + \int_{1-\alpha}^1 |t - 1 + \lambda(1-\alpha)| |f'(tb + (1-t)a)| dt \right] \\
& \leq (b-a) \left\{ \left(\int_0^{1-\alpha} |t - \alpha\lambda|^p dt \right)^{\frac{1}{p}} \left(\int_0^{1-\alpha} |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \right. \\
(2.18) \quad & \left. + \left(\int_{1-\alpha}^1 |t - 1 + \lambda(1-\alpha)|^p dt \right)^{\frac{1}{p}} \left(\int_{1-\alpha}^1 |f'(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Since $|f'|^q$ is convex on $[a, b]$, for $\alpha \in [0, 1)$ by the inequality (1.1), we get

$$\begin{aligned}
\int_0^{1-\alpha} |f'(tb + (1-t)a)|^q dt &= (1-\alpha) \left[\frac{1}{(1-\alpha)(b-a)} \int_a^{(1-\alpha)b + \alpha a} |f'(x)|^q dx \right] \\
(2.19) \quad &\leq (1-\alpha) \frac{|f'((1-\alpha)b + \alpha a)|^q + |f'(a)|^q}{2}.
\end{aligned}$$

The inequality (2.19) holds for $\alpha = 1$ too. Similarly, for $\alpha \in (0, 1]$ by the inequality (1.1), we have

$$\begin{aligned}
\int_{1-\alpha}^1 |f'(tb + (1-t)a)|^q dt &= \alpha \left[\frac{1}{\alpha(b-a)} \int_{(1-\alpha)b + \alpha a}^b |f'(x)|^q dx \right] \\
(2.20) \quad &\leq \alpha \frac{|f'((1-\alpha)b + \alpha a)|^q + |f'(b)|^q}{2}.
\end{aligned}$$

The inequality (2.20) holds for $\alpha = 0$ too. By simple computation

$$(2.21) \quad \int_0^{1-\alpha} |t - \alpha\lambda|^p dt = \begin{cases} \frac{(\alpha\lambda)^{p+1} + (1-\alpha-\alpha\lambda)^{p+1}}{p+1}, & \alpha\lambda \leq 1-\alpha \\ \frac{(\alpha\lambda)^{p+1} - (\alpha\lambda-1+\alpha)^{p+1}}{p+1}, & \alpha\lambda \geq 1-\alpha \end{cases},$$

and

$$(2.22) \quad \int_{1-\alpha}^1 |t - 1 + \lambda(1-\alpha)|^p dt = \begin{cases} \frac{[\lambda(1-\alpha)]^{p+1} + [\alpha - \lambda(1-\alpha)]^{p+1}}{p+1}, & 1-\alpha \leq 1-\lambda(1-\alpha) \\ \frac{[\lambda(1-\alpha)]^{p+1} - [\lambda(1-\alpha) - \alpha]^{p+1}}{p+1}, & 1-\alpha \geq 1-\lambda(1-\alpha) \end{cases},$$

thus, using (2.19)-(2.22) in (2.18), we obtain the inequality (2.16). This completes the proof. \square

Corollary 7. *Let the assumptions of Theorem 4 hold. Then for $\alpha = \frac{1}{2}$ and $\lambda = \frac{1}{3}$, from the inequality (2.16) we get the following Simpson type inequality*

$$(2.23) \quad \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq \frac{b-a}{12} \left(\frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \left\{ \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(a)|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \right\},$$

which is the same of the inequality (1.3).

Remark 1. *We note that if we use convexity of $|f'|^q$ in the inequality (2.23) then we obtain the inequality (1.2).*

Corollary 8. *Let the assumptions of Theorem 4 hold. Then for $\alpha = \frac{1}{2}$ and $\lambda = 0$, from the inequality (2.16) we get the following midpoint type inequality*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq \frac{b-a}{4} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(a)|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \right\}.$$

Corollary 9. *Let the assumptions of Theorem 4 hold. Then for $\alpha = \frac{1}{2}$ and $\lambda = 1$, from the inequality (2.16) we get the following trapezoid type inequality*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq \frac{b-a}{4} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(a)|^q}{2} \right)^{\frac{1}{q}} + \left(\frac{|f'(\frac{a+b}{2})|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \right\}.$$

Theorem 5. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I^\circ$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. If $|f'|^q$ is convex on $[a, b]$, $q > 1$, then the following inequality holds,*

$$(2.24) \quad \left| \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq (b-a)$$

$$\times \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ \begin{array}{ll} \left[\varepsilon_1^{\frac{1}{p}} \delta_3^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} \delta_4^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\alpha \leq 1-\lambda(1-\alpha) \\ \left[\varepsilon_1^{\frac{1}{p}} \delta_3^{\frac{1}{q}} + \varepsilon_4^{\frac{1}{p}} \delta_4^{\frac{1}{q}} \right], & \alpha\lambda \leq 1-\lambda(1-\alpha) \leq 1-\alpha \\ \left[\varepsilon_2^{\frac{1}{p}} \delta_3^{\frac{1}{q}} + \varepsilon_3^{\frac{1}{p}} \delta_4^{\frac{1}{q}} \right], & 1-\alpha \leq \alpha\lambda \leq 1-\lambda(1-\alpha) \end{array} \right.$$

where

$$\begin{aligned}\delta_3 &= \frac{|f'(b)|^q (1-\alpha)^2 + (1-\alpha^2) |f'(a)|^q dt}{2}, \quad \delta_4 = \frac{|f'(b)|^q \alpha (2-\alpha) + \alpha^2 |f'(a)|^q dt}{2}, \\ \varepsilon_1 &= (\alpha\lambda)^{p+1} + (1-\alpha-\alpha\lambda)^{p+1}, \quad \varepsilon_2 = (\alpha\lambda)^{p+1} - (\alpha\lambda - 1 + \alpha)^{p+1}, \\ \varepsilon_3 &= [\lambda(1-\alpha)]^{p+1} + [\alpha - \lambda(1-\alpha)]^{p+1}, \quad \varepsilon_4 = [\lambda(1-\alpha)]^{p+1} - [\lambda(1-\alpha) - \alpha]^{p+1}, \\ \text{and } \frac{1}{p} + \frac{1}{q} &= 1.\end{aligned}$$

Proof. From Lemma 2 and by Hölder's integral inequality, we have the inequality (2.18). Since $|f'|^q$ is convex on $[a, b]$, we know that for $t \in [0, 1-\alpha]$ and $t \in [1-\alpha, 1]$

$$|f'(tb + (1-t)a)|^q \leq t |f'(b)|^q + (1-t) |f'(a)|^q.$$

Hence

$$\begin{aligned}& \left| \lambda(\alpha f(a) + (1-\alpha)f(b)) + (1-\lambda)f(\alpha a + (1-\alpha)b) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq (b-a) \left\{ \left(\int_0^{1-\alpha} |t - \alpha\lambda|^p dt \right)^{\frac{1}{p}} \left(\int_0^{1-\alpha} t |f'(b)|^q + (1-t) |f'(a)|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_{1-\alpha}^1 |t - 1 + \lambda(1-\alpha)|^p dt \right)^{\frac{1}{p}} \left(\int_{1-\alpha}^1 t |f'(b)|^q + (1-t) |f'(a)|^q dt \right)^{\frac{1}{q}} \right\} \\ & \leq (b-a) \left\{ \left(\int_0^{1-\alpha} |t - \alpha\lambda|^p dt \right)^{\frac{1}{p}} \left(\frac{|f'(b)|^q (1-\alpha)^2 + (1-\alpha^2) |f'(a)|^q dt}{2} \right)^{\frac{1}{q}} \right. \\ (2.25) \quad & \left. + \left(\int_{1-\alpha}^1 |t - 1 + \lambda(1-\alpha)|^p dt \right)^{\frac{1}{p}} \left(\frac{|f'(b)|^q \alpha (2-\alpha) + \alpha^2 |f'(a)|^q dt}{2} \right)^{\frac{1}{q}} \right\}.\end{aligned}$$

thus, using (2.21), (2.22) in (2.25), we obtain the inequality (2.24). This completes the proof. \square

Corollary 10. *Let the assumptions of Theorem 5 hold. Then for $\alpha = \frac{1}{2}$ and $\lambda = \frac{1}{3}$, from the inequality (2.16) we get the following Simpson type inequality*

$$\begin{aligned}& \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{b-a}{12} \left(\frac{1+2^{p+1}}{3(p+1)} \right)^{\frac{1}{p}} \left\{ \left(\frac{3|f'(b)|^q + |f'(a)|^q}{4} \right)^{\frac{1}{q}} + \left(\frac{3|f'(a)|^q + |f'(b)|^q}{4} \right)^{\frac{1}{q}} \right\},\end{aligned}$$

which is the same of the inequality (1.2).

3. SOME APPLICATIONS FOR SPECIAL MEANS

Let us recall the following special means of arbitrary real numbers a, b with $a \neq b$ and $\alpha \in [0, 1]$:

- (1) The weighted arithmetic mean

$$A_\alpha(a, b) := \alpha a + (1 - \alpha)b, \quad a, b \in \mathbb{R}.$$

- (2) The unweighted arithmetic mean

$$A(a, b) := \frac{a + b}{2}, \quad a, b \in \mathbb{R}.$$

- (3) The weighted geometric mean

$$G_\alpha(a, b) = a^\alpha b^{1-\alpha}, \quad a, b > 0.$$

- (4) The unweighted geometric mean

$$G(a, b) = \sqrt{ab}, \quad a, b > 0.$$

- (5) The weighted harmonic mean

$$H_\alpha(a, b) := \left(\frac{\alpha}{a} + \frac{1 - \alpha}{b} \right)^{-1}, \quad a, b \in \mathbb{R} \setminus \{0\}.$$

- (6) The unweighted harmonic mean

$$H(a, b) := \frac{2ab}{a + b}, \quad a, b \in \mathbb{R} \setminus \{0\}.$$

- (7) The Logarithmic mean

$$L(a, b) := \frac{b - a}{\ln|b| - \ln|a|}, \quad |a| \neq |b|, \quad ab \neq 0.$$

- (8) Then n-Logarithmic mean

$$L_n(a, b) := \left(\frac{b^{n+1} - a^{n+1}}{(n+1)(b-a)} \right)^{\frac{1}{n}}, \quad n \in \mathbb{Z} \setminus \{-1, 0\}, \quad a, b \in \mathbb{R}, \quad a \neq b.$$

- (9) The identric mean

$$I(a, b) = \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}}, \quad a, b > 0, \quad a \neq b.$$

Proposition 1. Let $a, b \in \mathbb{R}$ with $a < b$, $0 \notin [a, b]$ and $n \in \mathbb{Z}$, $|n| \geq 2$. Then, for $\alpha, \lambda \in [0, 1]$ and $q \geq 1$, we have the following inequality:

$$|\lambda A_\alpha(a^n, b^n) + (1 - \lambda) A_\alpha^n(a, b) - L_n^n(a, b)| \leq \begin{cases} (b-a)|n| \left\{ \gamma_2^{1-\frac{1}{q}} \left(\mu_1 |b|^{(n-1)q} + \mu_2 |a|^{(n-1)q} \right)^{\frac{1}{q}} + v_2^{1-\frac{1}{q}} \left(\eta_3 |b|^{(n-1)q} + \eta_4 |a|^{(n-1)q} \right)^{\frac{1}{q}} \right\}, & \alpha\lambda \leq 1 - \alpha \leq 1 - \lambda(1 - \alpha) \\ (b-a)|n| \left\{ \gamma_2^{1-\frac{1}{q}} \left(\mu_1 |b|^{(n-1)q} + \mu_2 |a|^{(n-1)q} \right)^{\frac{1}{q}} + v_1^{1-\frac{1}{q}} \left(\eta_1 |b|^{(n-1)q} + \eta_2 |a|^{(n-1)q} \right)^{\frac{1}{q}} \right\}, & \alpha\lambda \leq 1 - \lambda(1 - \alpha) \leq 1 - \alpha, \\ (b-a)|n| \left\{ \gamma_1^{1-\frac{1}{q}} \left(\mu_3 |b|^{(n-1)q} + \mu_4 |a|^{(n-1)q} \right)^{\frac{1}{q}} + v_2^{1-\frac{1}{q}} \left(\eta_3 |b|^{(n-1)q} + \eta_4 |a|^{(n-1)q} \right)^{\frac{1}{q}} \right\}, & 1 - \alpha \leq \alpha\lambda \leq 1 - \lambda(1 - \alpha) \end{cases},$$

where $\gamma_1, \gamma_2, v_1, v_2, \mu_1, \mu_2, \mu_3, \mu_4, \eta_1, \eta_2, \eta_3, \eta_4$ numbers are defined as in (2.3)-(2.6).

Proof. The assertion follows from Theorem 3, for $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$, $|n| \geq 2$. \square

Proposition 2. Let $a, b \in \mathbb{R}$ with $a < b$, $0 \notin [a, b]$, and $n \in \mathbb{Z}$, $|n| \geq 2$. Then, for $\alpha, \lambda \in [0, 1]$ and $q > 1$, we have the following inequality:

$$|\lambda A_\alpha(a^n, b^n) + (1 - \lambda) A_\alpha^n(a, b) - L_n^n(a, b)| \leq (b-a) \left(\frac{1}{p+1} \right)^{\frac{1}{p}} |n| \times \begin{cases} \left[(1-\alpha)^{\frac{1}{q}} \varepsilon_1^{\frac{1}{p}} \theta_1 + \alpha^{\frac{1}{q}} \varepsilon_3^{\frac{1}{p}} \theta_2 \right], & \alpha\lambda \leq 1 - \alpha \leq 1 - \lambda(1 - \alpha) \\ \left[(1-\alpha)^{\frac{1}{q}} \varepsilon_1^{\frac{1}{p}} \theta_1 + \alpha^{\frac{1}{q}} \varepsilon_4^{\frac{1}{p}} \theta_2 \right], & \alpha\lambda \leq 1 - \lambda(1 - \alpha) \leq 1 - \alpha, \\ \left[(1-\alpha)^{\frac{1}{q}} \varepsilon_2^{\frac{1}{p}} \theta_1 + \alpha^{\frac{1}{q}} \varepsilon_3^{\frac{1}{p}} \theta_2 \right], & 1 - \alpha \leq \alpha\lambda \leq 1 - \lambda(1 - \alpha) \end{cases}$$

where

$$\theta_1 = A^{\frac{1}{q}} \left(\left| A_\alpha^{(n-1)q}(a, b) \right|, |a|^{(n-1)q} \right), \theta_2 = A^{\frac{1}{q}} \left(\left| A_\alpha^{(n-1)q}(a, b) \right|, |b|^{(n-1)q} \right), \frac{1}{p} + \frac{1}{q} = 1,$$

and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ numbers are defined as in (2.17).

Proof. The assertion follows from Theorem 4, for $f(x) = x^n$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$, $|n| \geq 2$. \square

Proposition 3. Let $a, b \in \mathbb{R}$ with $a < b$, $0 \notin [a, b]$. Then, for $\alpha, \lambda \in [0, 1]$ and $q \geq 1$, we have the following inequality:

$$\begin{aligned} & |\lambda H_\alpha^{-1}(a, b) + (1 - \lambda) A_\alpha^{-1}(a, b) - L^{-1}(a, b)| \\ & \leq \begin{cases} (b - a) \left\{ \gamma_2^{1-\frac{1}{q}} \left(\mu_1 \frac{1}{|b|^{2q}} + \mu_2 \frac{1}{|a|^{2q}} \right)^{\frac{1}{q}} \right. \\ \quad \left. + v_2^{1-\frac{1}{q}} \left(\eta_3 \frac{1}{|b|^{2q}} + \eta_4 \frac{1}{|a|^{2q}} \right)^{\frac{1}{q}} \right\}, & \alpha\lambda \leq 1 - \alpha \leq 1 - \lambda(1 - \alpha) \\ (b - a) \left\{ \gamma_2^{1-\frac{1}{q}} \left(\mu_1 \frac{1}{|b|^{2q}} + \mu_2 \frac{1}{|a|^{2q}} \right)^{\frac{1}{q}} \right. \\ \quad \left. + v_1^{1-\frac{1}{q}} \left(\eta_1 \frac{1}{|b|^{2q}} + \eta_2 \frac{1}{|a|^{2q}} \right)^{\frac{1}{q}} \right\}, & \alpha\lambda \leq 1 - \lambda(1 - \alpha) \leq 1 - \alpha, \\ (b - a) \left\{ \gamma_1^{1-\frac{1}{q}} \left(\mu_3 \frac{1}{|b|^{2q}} + \mu_4 \frac{1}{|a|^{2q}} \right)^{\frac{1}{q}} \right. \\ \quad \left. + v_2^{1-\frac{1}{q}} \left(\eta_3 \frac{1}{|b|^{2q}} + \eta_4 \frac{1}{|a|^{2q}} \right)^{\frac{1}{q}} \right\}, & 1 - \alpha \leq \alpha\lambda \leq 1 - \lambda(1 - \alpha) \end{cases}, \end{aligned}$$

where $\gamma_1, \gamma_2, v_1, v_2, \mu_1, \mu_2, \mu_3, \mu_4, \eta_1, \eta_2, \eta_3, \eta_4$ numbers are defined as in (2.3)-(2.6).

Proof. The assertion follows from Theorem 3, for $f(x) = \frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$. \square

Proposition 4. Let $a, b \in \mathbb{R}$ with $0 < a < b$. Then, for $\alpha, \lambda \in [0, 1]$ and $q > 1$, we have the following inequality:

$$\begin{aligned} & |\lambda H_\alpha^{-1}(a, b) + (1 - \lambda) A_\alpha^{-1}(a, b) - L^{-1}(a, b)| \leq (b - a) \left(\frac{1}{p + 1} \right)^{\frac{1}{p}} \\ & \times \begin{cases} \left[(1 - \alpha)^{\frac{1}{q}} \varepsilon_1^{\frac{1}{p}} \theta_3 + \alpha^{\frac{1}{q}} \varepsilon_3^{\frac{1}{p}} \theta_4 \right], & \alpha\lambda \leq 1 - \alpha \leq 1 - \lambda(1 - \alpha) \\ \left[(1 - \alpha)^{\frac{1}{q}} \varepsilon_1^{\frac{1}{p}} \theta_3 + \alpha^{\frac{1}{q}} \varepsilon_4^{\frac{1}{p}} \theta_4 \right], & \alpha\lambda \leq 1 - \lambda(1 - \alpha) \leq 1 - \alpha, \\ \left[(1 - \alpha)^{\frac{1}{q}} \varepsilon_2^{\frac{1}{p}} \theta_3 + \alpha^{\frac{1}{q}} \varepsilon_3^{\frac{1}{p}} \theta_4 \right], & 1 - \alpha \leq \alpha\lambda \leq 1 - \lambda(1 - \alpha) \end{cases} \end{aligned}$$

where

$$\theta_3 = H^{-\frac{1}{q}}(A_\alpha^{2q}(a, b), a^{2q}), \theta_4 = H^{-\frac{1}{q}}(A_\alpha^{2q}(a, b), b^{2q}), \frac{1}{p} + \frac{1}{q} = 1,$$

and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ numbers are defined as in (2.17).

Proof. The assertion follows from Theorem 4, for $f(x) = \frac{1}{x}$, $x \in \mathbb{R} \setminus \{0\}$. \square

Proposition 5. Let $a, b \in \mathbb{R}$ with $0 < a < b$. Then, for $\alpha, \lambda \in [0, 1]$ and $q \geq 1$, we have the following inequality:

$$|A_\lambda(\ln G_\alpha(a, b), \ln A_\alpha(a, b)) - \ln I(a, b)| \leq \begin{cases} (b-a) \left\{ \gamma_2^{1-\frac{1}{q}} \left(\mu_1 \frac{1}{b^q} + \mu_2 \frac{1}{a^q} \right)^{\frac{1}{q}} + v_2^{1-\frac{1}{q}} \left(\eta_3 \frac{1}{b^q} + \eta_4 \frac{1}{a^q} \right)^{\frac{1}{q}} \right\}, & \alpha\lambda \leq 1 - \alpha \leq 1 - \lambda(1 - \alpha) \\ (b-a) \left\{ \gamma_2^{1-\frac{1}{q}} \left(\mu_1 \frac{1}{b^q} + \mu_2 \frac{1}{a^q} \right)^{\frac{1}{q}} + v_1^{1-\frac{1}{q}} \left(\eta_1 \frac{1}{b^q} + \eta_2 \frac{1}{a^q} \right)^{\frac{1}{q}} \right\}, & \alpha\lambda \leq 1 - \lambda(1 - \alpha) \leq 1 - \alpha \\ (b-a) \left\{ \gamma_1^{1-\frac{1}{q}} \left(\mu_3 \frac{1}{b^q} + \mu_4 \frac{1}{a^q} \right)^{\frac{1}{q}} + v_2^{1-\frac{1}{q}} \left(\eta_3 \frac{1}{b^q} + \eta_4 \frac{1}{a^q} \right)^{\frac{1}{q}} \right\}, & 1 - \alpha \leq \alpha\lambda \leq 1 - \lambda(1 - \alpha) \end{cases}$$

where $\gamma_1, \gamma_2, v_1, v_2, \mu_1, \mu_2, \mu_3, \mu_4, \eta_1, \eta_2, \eta_3, \eta_4$ numbers are defined as in (2.3)-(2.6).

Proof. The assertion follows from Theorem 3, for $f(x) = -\ln x, x > 0$. \square

Proposition 6. Let $a, b \in \mathbb{R}$ with $0 < a < b$. Then, for $\alpha, \lambda \in [0, 1]$ and $q > 1$, we have the following inequality:

$$|A_\lambda(\ln G_\alpha(a, b), \ln A_\alpha(a, b)) - \ln I(a, b)| \leq (b-a) \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \times \begin{cases} \left[(1-\alpha)^{\frac{1}{q}} \varepsilon_1^{\frac{1}{p}} \theta_3 + \alpha^{\frac{1}{q}} \varepsilon_3^{\frac{1}{p}} \theta_4 \right], & \alpha\lambda \leq 1 - \alpha \leq 1 - \lambda(1 - \alpha) \\ \left[(1-\alpha)^{\frac{1}{q}} \varepsilon_1^{\frac{1}{p}} \theta_3 + \alpha^{\frac{1}{q}} \varepsilon_4^{\frac{1}{p}} \theta_4 \right], & \alpha\lambda \leq 1 - \lambda(1 - \alpha) \leq 1 - \alpha \\ \left[(1-\alpha)^{\frac{1}{q}} \varepsilon_2^{\frac{1}{p}} \theta_3 + \alpha^{\frac{1}{q}} \varepsilon_3^{\frac{1}{p}} \theta_4 \right], & 1 - \alpha \leq \alpha\lambda \leq 1 - \lambda(1 - \alpha) \end{cases},$$

where

$$\theta_3 = H^{-\frac{1}{q}}(A_\alpha^q(a, b), a^q), \theta_4 = H^{-\frac{1}{q}}(A_\alpha^q(a, b), b^q), \frac{1}{p} + \frac{1}{q} = 1,$$

and $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ numbers are defined as in (2.17).

Proof. The assertion follows from Theorem 4, for $f(x) = -\ln x, x > 0$. \square

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